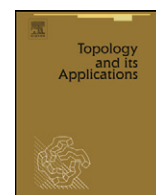



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Critical points of higher order for the normal map of immersions in \mathbb{R}^d

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ABSTRACT

We study the critical points of the normal map $\nu : NM \rightarrow \mathbb{R}^{k+n}$, where M is an immersed k -dimensional submanifold of \mathbb{R}^{k+n} , NM is the normal bundle of M and $\nu(m, u) = m + u$ if $u \in N_m M$. Usually, the image of these critical points is called the focal set. However, in that set there is a subset where the focusing is highest, as happens in the case of curves in \mathbb{R}^3 with the curve of the centers of spheres with contact of third order with the curve. We give a definition of r -critical points of a smooth map between manifolds, and apply it to study the 2 and 3-critical points of the normal map in general and the 2-critical points for the case $k = n = 2$ in detail. In the later case we analyze the relation with the strong principal directions of Montaldi (1986) [2].

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1. Introduction

Classically the focal set of a differential submanifold is given through the analysis of the singularities of the family of distance squared functions over the submanifold, see [4]. J. Montaldi characterized in [2] the singularities of corank 2 of distance squared functions on surfaces immersed in \mathbb{R}^4 as semiumbilic points.

Also, if we consider a differentiable k -dimensional manifold M immersed in \mathbb{R}^{k+n} , we know that its focal set can also be interpreted as the image of the critical points of the *normal map* $\nu(m, u) : NM \rightarrow \mathbb{R}^{k+n}$ defined by $\nu(m, u) = \pi_N(m, u) + u$, for $u \in N_m M$, where $\pi_N : NM \rightarrow M$ denotes the normal bundle.

On the other hand, the concept of curvature ellipse at a point of a surface M immersed in \mathbb{R}^4 was treated with full details in [1]. It is defined as the locus of all the end points of the curvature vectors of the normal sections along all the tangent directions to M at a point in it. This ellipse lies in the normal subspace of that point and it is completely determined by the second fundamental. We call *Veronese of curvature* to the natural generalization of the curvature ellipse for higher dimensions of M .

In this work, we describe first the focal set and its geometrical relation to the Veronese of curvature for k -dimensional immersions in \mathbb{R}^{k+n} . Then we define the r -critical points of a differential application $f : H \rightarrow K$ between two differential manifolds and characterize the 2 and 3-critical points of the normal map. The number of these critical points at $m \in M$ may

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depend on the degenerations of the curvature ellipse and we calculate those numbers in the particular case that M is an immersed surface in \mathbb{R}^4 .

2. The Veronese of curvature

In the following, M will be a smooth k -dimensional manifold immersed in \mathbb{R}^{k+n} . However, since all our study will be local, we shall suppose without loss of generality that M is a regular submanifold of \mathbb{R}^{k+n} . Over M we have the tangent bundle $\pi : TM \rightarrow M$, and the normal bundle $\pi_N : NM \rightarrow M$. Their fibers over $m \in M$ will be denoted by $T_m M$ and $N_m M = (T_m M)^\perp$, respectively. We will denote by $\mathfrak{X}(M)$ the Lie algebra of smooth vector fields, and if E is the total space of a vector bundle over M , $\Gamma(E)$ will be the module of its smooth sections. The value of such a section S at $m \in M$ will be denoted S_m .

The usual inner product will be denoted by a dot and the first fundamental form of M by $g \in \Gamma(T^*M \otimes T^*M)$. If $X \in T_m \mathbb{R}^{k+n}$, we will have the decomposition $X = X^\top + X^\perp$, with $X^\top \in T_m M$, $X^\perp \in N_m M$, and D_X will stand for the ordinary directional derivative. All vectors of $T_m M$ and of $N_m M$ are considered as elements of $T_m \mathbb{R}^{k+n}$ and frequently identified as usual with points of \mathbb{R}^{k+n} .

The second fundamental form will be denoted $\alpha \in \Gamma(NM \otimes T^*M \otimes T^*M)$. If $X, Y \in \mathfrak{X}(M)$ and $Z \in \Gamma(NM)$, we have $Z \cdot \alpha(X, Y) = Z \cdot D_X Y = Z \cdot D_Y X$. Related to α is the tensor field $\mathcal{A} \in \Gamma(TM \otimes N^*M \otimes T^*M)$ given by $\mathcal{A}_Z(X) = (D_X Z)^\top$. Then, $Z \cdot \alpha(X, Y) = -Y \cdot \mathcal{A}_Z(X)$.

Let $PT_m M$ be the projective space of directions in $T_m M$. The second fundamental form defines a map $\eta_m : PT_m M \rightarrow N_m M$, which we call the *Veronese of curvature*, by

$$\eta_m([t]) = \eta_m(t) = \frac{\alpha_m(t, t)}{t \cdot t}, \quad t \in T_m M \setminus \{0\}.$$

3. The focal set

We will describe in this section and the next the relations between the Veronese of curvature and the focal set of the immersion M in \mathbb{R}^{k+n} .

A useful definition of the focal set of M goes as follows.

Definition 3.1. Let ν be the normal map of M . Then the *focal set of M* , denoted here by $\mathcal{F}(M)$, is defined as the set of critical points of ν . Since there will be little risk of confusion, the same name will be used for the image of $\mathcal{F}(M)$ by ν .

The next result is well known:

Proposition 3.2. *The focal set of M is given by*

$$\mathcal{F}(M) = \{(m, u) \in NM : \det(g_m - u \cdot \alpha_m) = 0, \text{ where } m = \pi_N(m, u)\},$$

where the determinant can be computed by means of any orthonormal basis of $T_m M$.

Proof. Let w_i , $i = 1, \dots, n$, be a local orthonormal frame of NM in a neighborhood U of $m \in M$ and let t_i , $i = 1, \dots, k$, be an orthonormal frame of TM in U . By means of the first of those frames we can work with a trivialization of NM on U given by $u \approx (\pi_N(m, u), x_1, \dots, x_n)$, where the x_i are such that $u = \sum_i x_i w_{im}$, being $m = \pi_N(m, u)$. Thus the map ν can be expressed as $\nu(m, x_1, \dots, x_n) = m + \sum_i x_i w_{im}$.

If $X \in T_m M$, then $d\nu(X, 0) = X + \sum_i x_i dw_i(X) = X + \sum_i x_i D_X w_i$, and $d\nu(0, \partial_{x_i}) = w_i$. Since the w_i are orthonormal, the vanishing of $\det(d\nu)$ is equivalent to the vanishing of the determinant of the orthogonal projection of $d\nu|_{T_m M}$ into $T_m M$, that is to the vanishing of the determinant of the endomorphism of $T_m M$ given by $X \rightarrow (d\nu(X, 0))^\top = X + \sum_i x_i (D_X w_i)^\top$. The component in t_{jm} of $(d\nu(t_{im}, 0))^\top$ is $t_{im} \cdot t_{jm} + \sum_b x_b (D_{t_{im}} w_b) \cdot t_{jm} = g_m(t_{im}, t_{jm}) - u_m \cdot \alpha_m(t_{im}, t_{jm})$, where $u = \sum_b x_b w_{bm}$. That is, the condition is equivalent to the vanishing at m of the matrix with coefficients $(g - u \cdot \alpha)(t_i, t_j)$. \square

Proposition 3.3. *Let $\mathcal{F}_m(M) = \mathcal{F}(M) \cap N_m M$. Then the following properties are satisfied:*

1. If $u : M \rightarrow NM$ be a local section in a neighborhood of $m \in M$, then $\det(g_m - u_m \cdot \alpha_m) = \det(d(\nu \circ u)_m^\top)$.
2. If $u \in \mathcal{F}_m(M)$, then there exists $t \in T_m M \setminus \{0\}$, such that $g_m(t) = u \cdot \alpha_m(t) \in T_m^* M$. In this case, we say that u belongs to the focal set over t . In the following items, t and u satisfy that property.
3. $u \cdot \eta_m(t) = 1$. In particular, $u \neq 0$, $\eta_m(t) \neq 0$.
4. $\eta_m(t) \notin (d\eta_m)(T_t T_m M) \subset T_{\eta_m(t)} N_m M$, under the usual identification of $N_m M$ with $T_{\eta_m(t)} N_m M$.

Proof. 1) If, as before, we take a local frame w_i of NM , we can write $u = \sum_i u^i w_i$. Thus, $\nu \circ u = \text{id} + \sum_i u^i w_i$, whence if $X \in T_m M$, we will have $d(\nu \circ u)_m^\top(X) = (X + \sum_i ((D_X u^i) w_{im} + u^i D_X w_i))^\top = X + A_{u_m}(X)$. Therefore,

$$\det d(\zeta \circ u)_m^\top = \det(d(\nu \circ u)(t_a) \cdot t_b) = \det(\delta_{ab} - u_m \cdot \alpha_m(t_a, t_b)).$$

2) We can look at $g_m - u \cdot \alpha_m$ as a linear map from $T_m M$ to $T_m^* M$. Since it maps linearly a vector space into another of same dimension and has zero determinant, we conclude that there is some non-zero $t \in T_m M$ such that $(g_m - u \cdot \alpha_m)(t) = 0$.

3) If the 1-form $(g_m - u \cdot \alpha_m)(t)$ acts upon the vector t itself, we get $t \cdot t - u \cdot \alpha_m(t, t) = 0$, whence, by dividing by $t \cdot t \neq 0$, we obtain the claim.

4) Note that here η_m is taken as a map from $T_m M \setminus \{0\}$ to $N_m M$. For making the calculations easier we can assume that $t \cdot t = 1$. Then, if $X \in T_t T_m M$, we have

$$d\eta_m(X) = \frac{2}{(t \cdot t)^2} ((t \cdot t)\alpha_m(t, X) - (t \cdot X)\alpha_m(t, t)) = 2(\alpha_m(t, X) - (t \cdot X)\eta_m(t)).$$

Suppose that this is equal to $\eta_m(t)$. By inner multiplication of this with u we get $u \cdot \eta_m(t) = t \cdot t = 1$, while the same multiplication with $d\eta_m(X)$ yields $2(u \cdot \alpha_m(t) - g_m(t))(X) = 0$, which is absurd. \square

In general, $\mathcal{F}(M)$ will be a hypersurface of NM , possibly with singularities, whose intersection with each fiber $N_m M$ will be an algebraic hypersurface of degree k . Thus, in the case of a surface $M \subset \mathbb{R}^{2+n}$, the intersection $\mathcal{F}_m(M)$ of $\mathcal{F}(M)$ with $N_m M$ will be a quadric.

4. Focal set and the inverted pedal of the Veronese of curvature

Definition 4.1. Let P be a smooth manifold and $\mu : P \rightarrow \mathbb{R}^n$ a smooth map. For each $p \in P$, let $\text{ped}_\mu(p)$ be the nearest point to the origin among those of the affine subspace tangent to $\mu(P)$ at $\mu(p)$, i.e. $\{\mu(p) + d\mu(X) : X \in T_p P\}$. The resulting map $\text{ped}_\mu : P \rightarrow \mathbb{R}^n$ is called the *pedal map* of μ . Let $\tilde{P}_\mu = \{p \in P : \text{ped}_\mu(p) \neq 0\}$. If $R : \mathbb{R}^n \setminus \{0\} \rightarrow \mathbb{R}^n \setminus \{0\}$ is the inversion with respect to the hypersphere with center 0 and unit radius, the composition $R \circ \text{ped}_\mu : \tilde{P}_\mu \rightarrow \mathbb{R}^n \setminus \{0\}$ (and sometimes, also its image) will be called the *inverted pedal* of μ .

Let us show the relation between the focal set of and the inverted pedal of η at $m \in M$. We are here interested solely in the study of $\mathcal{F}_m(M)$. This justifies the use of the following simplified notation in this section:

$$T = T_m M \setminus \{0\}, \quad N = N_m M, \quad \alpha = \alpha_m, \quad \eta = \eta_m, \quad \mathcal{F} = \mathcal{F}_m(M), \quad g = g_m.$$

Proposition 4.2. Let $z \in N$ be a point in the inverted pedal of η . Then $z \in \mathcal{F}$.

Proof. Note that here η is taken as a map from T to N . Let $t \in T$ and let $0 \neq z = \eta(t) + d\eta(X)$, with $X \in T_t(T)$, a point in the pedal of η so that $R(z)$ belongs to the inverted pedal of η . We must have $z \cdot d\eta(T_t T) = d(z \cdot \eta)(T_t T) = 0$. In particular, $z \cdot d\eta(X) = 0$, whence $z \cdot z = z \cdot \eta(t) \neq 0$. Also, $d(z \cdot \eta)_t = 0$. Hence t is a critical point of the map $t \mapsto z \cdot \eta(t)$. But one sees easily that this entails

$$g(t, t)z \cdot \alpha(t, t') - (t \cdot t')z \cdot \alpha(t, t) = 0$$

for any $t' \in T_t T$, i.e. $(z \cdot \alpha - z \cdot \eta(t)g)(t) = 0$, and this requires the vanishing of $\det(z \cdot \alpha - z \cdot \eta(t)g)$. By dividing that determinant by $(-z \cdot \eta(t))^k$, we conclude that

$$\det\left(g - \frac{z}{z \cdot \eta(t)} \cdot \alpha\right) = 0,$$

that is

$$\frac{z}{z \cdot \eta(t)} = \frac{z}{z \cdot z} = R(z) \in \mathcal{F}. \quad \square$$

Let us see whether there is some form of converse of this. We put $B_t = R(\text{ped}_\eta(t)) + \alpha(t, T)^\perp$. Thus, B_t is an affine subspace of N passing by $R(\text{ped}_\eta(t))$. Note that $\alpha(t, T)$ is the vector space generated by $\eta(t)$ and the tangent space of the Veronese of curvature at $\eta(t)$. Let $\tilde{T} = \{t \in T : \text{ped}_\eta(t) \neq 0\}$.

Theorem 4.3. \mathcal{F} is the union of the inverted pedal of η with $\bigcup_{t \in \tilde{T}} B_t$.

Proof. Let $x \in N$ be a point in \mathcal{F} . Then $\det(g - x \cdot \alpha) = 0$. Let $t \in T$, $t \cdot t = 1$, be such that $g(t) = x \cdot \alpha(t)$. We know that then $x \cdot \eta(t) = 1$ and $\eta(t) \notin (d\eta)(T_t T)$. Let $z = \text{ped}_\eta(t)$; if $z = 0$ we would have $\eta(t) \in (d\eta_p)(T_t T)$, which is absurd. As we have seen before we will have $(z \cdot z)g(t) = z \cdot \alpha(t)$, from which we obtain $g(t) = R(z) \cdot \alpha(t)$. Therefore $(R(z) - x) \cdot \alpha(t) = 0$, that is $R(z) - x \in B_t$. Hence we can write this in the form

$$x = R(z) + u, \quad u \in \alpha(t, T)^\perp. \quad \square$$

This describes completely $\mathcal{F}(M)$. Note that $\dim \alpha(t, T) \leq k$. Hence, if for example $k = 2$ (M is thus a surface) and $n = 2$ then generically the dimension of \mathcal{F}_m will be that of $\eta(PT_m M)$, that is one; thus, \mathcal{F}_p will be a conic. If $n = 3$, it will be generically a ruled quadric surface.

5. r -Critical points of smooth maps

In this section we recall first the well-known notion of r -tangent bundle and define r -critical points. We shall use the letter π to denote the natural map $\pi : TM \rightarrow M$ for any smooth manifold M .

The r -tangent bundle $\pi^r : T^r M \rightarrow M$, for any non-negative integer r , is defined recursively as follows: the 0-tangent bundle is $\text{id} : M \rightarrow M$, the 1-tangent bundle is the tangent bundle. Suppose that we have defined $\pi^r : T^r M \rightarrow M$ and also a bundle $\pi_{r-1}^r : T^r M \rightarrow T^{r-1} M$ such that $\pi^r = \pi_{r-1}^r \circ \pi_{r-1}^r$, where $\pi = \pi_0^r$. Then we define the next total space

$$T^{r+1} M = \{X \in T(T^r M) : \pi(X) = d\pi_{r-1}^r(X)\}$$

and the maps $\pi^{r+1} = \pi^r \circ \pi$ and $\pi_{r-1}^{r+1}(X) = \pi(X) = d\pi_{r-1}^r(X)$. It is easy to see that $T^r M$ is the bundle of r -jets of curves in M . That is, if $X \in T^r M$, there is a curve $\gamma : I \rightarrow M$, where I is an open neighborhood of $0 \in \mathbb{R}$, such that $\gamma(0) = m$, and X stores all of the information of $\gamma(0), \gamma'(0), \dots, \gamma^{(r)}(0)$. In fact, assume that $\gamma^{(r)}$ is a smooth map from I to $T^r M$ and that $\pi_{r-1}^r \circ \gamma^{(r)} = \gamma^{(r-1)}$. Then, define $\gamma^{(r+1)} = \gamma^{(r)'} = d\gamma^{(r)} \circ \mathbf{1} : I \rightarrow T^{r+1} M$, where $\mathbf{1} : \mathbb{R} \rightarrow T\mathbb{R}$ is the unit vector field. Therefore

$$\pi(\gamma^{(r+1)}) = (\pi \circ d\gamma^{(r)} \circ \mathbf{1}) = \gamma^{(r)}$$

and $d\pi_{r-1}^r \circ \gamma^{(r+1)} = d(\pi_{r-1}^r \circ \gamma^{(r)}) \circ \mathbf{1} = d\gamma^{(r-1)} \circ \mathbf{1} = \gamma^{(r-1)'} = \gamma^{(r)}$. Hence, $\gamma^{(r+1)}(0) \in T^{r+1} M$.

If $A \in T^r M$, we shall denote by $T_A^{r+1} M$ the fiber of π_{r-1}^{r+1} over A . Let $VT^r M = \{X \in T(T^r M) : d\pi_{r-1}^r(X) = 0\}$ be the vertical bundle over $T^r M$, which is a vector bundle whose fibre upon $X \in T^r M$ is denoted $V_X T^r M$. Let $A \in T^r M$ and $X, Y \in T_A^r M$. Then $d\pi_{r-1}^r(X - Y) = d\pi_{r-1}^r(X) - d\pi_{r-1}^r(Y) = \pi(X) - \pi(Y) = 0$. Hence, $X - Y \in V_A T^r M$. Therefore the fibre $T_A^{r+1} M$ is an affine space. Its dimension is that of M . We define recursively the subset $O^r M \subset T^r M$ as follows. First, $O^1 M$ is the image of the zero section of $\pi : TM \rightarrow M$, i.e. $O_m^0 = \pi(O_m^1)$. Assume that $r > 1$ and that $O^{r-1} M$ has been defined so that the intersection of $O^{r-1} M$ with the fibre $T_m^{r-1} M$, for $m \in M$, is exactly one point denoted $O_m^{r-1} M$ and that $\pi_{r-2}^{r-1}(O_m^{r-1} M) = O_m^{r-2} M$. Then, $O_m^r M$ is by definition the zero of the vector space $T_{O_m^{r-1} M} T^r M$. Note that $\pi(O_m^r M) = O_m^{r-1} M$ and that $d\pi_{r-2}^{r-1}(O_m^r M) = \{0 \in T_{O_m^{r-2} M}(T^{r-1} M)\} = O_m^{r-1} M$. Hence, $O_m^r M \in T^r M$ and $\pi_{r-1}^r(O_m^r M) = O_m^{r-1} M$.

Finally, if $f : M \rightarrow N$ is a smooth map between manifolds, it induces a smooth map $f^{(r)} : T^r M \rightarrow T^r N$ given by $f^{(r)}(\gamma^{(r)}(0)) = (f \circ \gamma)^{(r)}(0)$ for any smooth curve $\gamma : I \rightarrow M$, where $0 \in I$. If $g : N \rightarrow P$ is another smooth map, then $(g \circ f)^{(r)} = g^{(r)} \circ f^{(r)}$.

Definition 5.1. Let $f : H \rightarrow K$ be a smooth map between manifolds. We say that $X \in T^1 H = TH$ is a 1-critical point of f if $X \neq 0$ and $f^{(1)}(X) \in O^1 K$ that is if $df(X) = 0$. And if $A \in T^r H$ we say that it is an r -critical point of f if $\pi_{r-1}^r(A)$ is an $(r-1)$ -critical point of f and $f^{(r)}(A) \in O^r K$.

6. 2-Critical points of ν for immersions in \mathbb{R}^{k+n}

Now, we will study the 2-critical points of the normal map $\nu : NM \rightarrow \mathbb{R}^{k+n}$. We use the following notations for the different bundles that we consider: $\pi : TM \rightarrow M$, $\pi_N : NM \rightarrow M$, $\pi_1 : T(NM) \rightarrow NM$, $\pi_T : T(T(NM)) \rightarrow T(NM)$, $\pi_2 : T^2(NM) \rightarrow T(NM)$.

Let $i^*(T\mathbb{R}^{k+n}) \rightarrow M$ be the induced bundle of the bundle $T\mathbb{R}^{k+n} \rightarrow \mathbb{R}^{k+n}$, by the inclusion $i : M \rightarrow \mathbb{R}^{k+n}$. The sections of this bundle are differentiable applications of the form $Z : M \rightarrow \mathbb{R}^{k+n}$. We can decompose each application in a unique way in two smooth summands $Z = Z^\top + Z^\perp$, where $Z^\top \in \mathfrak{X}(M)$ and $Z^\perp \in \Gamma(NM)$.

We define the operator $\nabla : \mathfrak{X}(M) \times \Gamma(i^*(T\mathbb{R}^{k+n})) \rightarrow \Gamma(i^*(T\mathbb{R}^{k+n}))$ putting $\nabla_X Z = (D_X Z^\top)^\top + (D_X Z^\perp)^\perp$, where $(D_X Z)_m = dZ(X_m)$ with the usual identification of $T_{Z_m} \mathbb{R}^{k+n}$ with \mathbb{R}^{k+n} . One can verify easily that ∇ is a linear connection in the vector bundle $i^*(T\mathbb{R}^{k+n}) \rightarrow M$, which preserves the dot product.

Also we can decompose the dual subbundle of $i^*(T\mathbb{R}^{k+n}) \rightarrow M$, in two subbundles, respectively isomorphic to the dual of the bundle TM , denoted T^*M , and to the dual of NM , denoted N^*M . This means, among other things, that we can look to tensor fields such as g or α as tensor fields over the vector bundle $i^*(T\mathbb{R}^{k+n})$. That is, if for instance $X, Y \in \Gamma(i^*(T\mathbb{R}^{k+n}))$ and $\beta \in \Gamma((i^*(T\mathbb{R}^{k+n}))^*)$ we will have

$$\beta(\alpha(X, Y)) := \beta(\alpha(X^\top, Y^\top)), \quad g(X, Y) := g(X^\top, Y^\top).$$

Then, we extend ∇ to sections of those subbundles in the usual manner, and to tensor fields that are sections of the tensor product of copies of the bundles TM , T^*M , NM and N^*M , that can be seen as a section of the tensorial product of copies of $i^*(T\mathbb{R}^{k+n}) \rightarrow M$ by its dual. This extension of ∇ will preserve also the dot product. For instance, if $W \in \mathfrak{X}(M)$ we have

$$\begin{aligned} D_W(Z \cdot \alpha(X, Y)) &= W(Z \cdot \alpha(X, Y)) = \nabla_W(Z \cdot \alpha(X, Y)) = (\nabla_W Z) \cdot \alpha(X, Y) \\ &\quad + Z \cdot (\nabla_W \alpha)(X, Y) + Z \cdot \alpha(\nabla_W X, Y) + Z \cdot \alpha(X, \nabla_W Y). \end{aligned}$$

The connection ∇ is determined by a metric connection in the principal fiber bundle of adapted frames of $i^*(T\mathbb{R}^{k+n}) \rightarrow M$, that is frames as $(t_1, \dots, t_k, w_1, \dots, w_n)$ where (t_1, \dots, t_k) is a frame of TM and (w_1, \dots, w_n) is one of NM .

Definition 6.1. Let $m \in M$, $v \in T_m M$ and $U \in N_m M$. In the following we will say that *there is a 2-critical point of v over (v, U)* , or that (v, U) *admits a 2-critical point of v* if there is some 2-critical point $X^2 \in T^2(NM)$ of v such that $U = \pi_2(X^2)$ and $v = (\pi_1^2 \circ \pi_N^{(2)})(X^2)$. In other terms, if $X^2 = u''(0)$, being $u : I \rightarrow NM$ a smooth curve, then $v = (\pi_N \circ u)'(0)$ and $U = u(0)$. In the same manner we say that $v \in T_m M$ *admits a 2-critical point of v* if there is $U \in N_m M$ such that (v, U) admits a 2-critical point of v .

The following theorem characterizes the properties that must have a pair (v, U) as above in order to admit a 2-critical point of v . In it we shall suppress the subindex m , that means evaluation at m , whenever this will not cause confusion, for instance when there appears any of the symbols U or v .

Theorem 6.2. Let $m \in M$, $v \in T_m M$ and $U \in N_m M$. Then, (v, U) admits a 2-critical point iff the following conditions are satisfied:

- $v \neq 0$ and $g(v, \cdot) - U \cdot \alpha(v, \cdot) = 0$, i.e. U belongs to the focal set over v .
- $U \cdot (\nabla_v \alpha)(v, v) = 0$.
- Let (t_1, \dots, t_k) be an orthonormal basis of $T_m M$ such that t_1 and v are parallel. Then the following linear system, whose unknowns are the components of $x = x^2 t_2 + \dots + x^k t_k$, has a solution:

$$g(x, t_j) - U \cdot \alpha(x, t_j) = U \cdot (\nabla_v \alpha)(v, t_j), \quad j = 2, \dots, k.$$

Proof. Suppose that $X^2 \in T^2(NM)$ can be written as $X^2 = \tilde{u}''(0)$, where $\tilde{u} : I \rightarrow NM$ is a smooth curve such that, if $\gamma := \pi_N \circ \tilde{u}$, we have $\tilde{u}(0) = U$ and $\gamma'(0) = v$. We have $\pi_2 \circ \tilde{u}'' = \tilde{u}'$ and $\pi_N \circ \tilde{u}' = \tilde{u}$. Let $X := \pi_2(X^2)$. For $t \in I$, $\tilde{u}(t) \in N_{\gamma(t)} N \subset \mathbb{R}^{k+n}$. Hence we may think also of \tilde{u} as a map from I to \mathbb{R}^{k+n} , and this justifies the use of the following notations:

- A will denote $X^2 = \tilde{u}''(0)$ as element of \mathbb{R}^{k+n} .
- V will denote $\pi_1^2(X^2) = \tilde{u}'(0)$ as element of \mathbb{R}^{k+n} .
- a will denote $\pi_N^{(2)}(X^2) = \gamma''(0)$ as element of \mathbb{R}^{k+n} .

This is taken in the same sense as when one speaks of the acceleration of a moving particle as a vector in \mathbb{R}^3 , without telling that it is the acceleration of the particle at the point m and velocity v .

Suppose that X^2 is 2-critical. Then X must be 1-critical, that is

$$dv(X) = dv(\tilde{u}'(0)) = (v \circ \tilde{u})'(0) = (\gamma + \tilde{u})'(0) = v + V = 0.$$

Since X cannot vanish we have $V \neq 0$, whence $v = \gamma'(0) \neq 0$. This means that we may suppose that $\gamma : I \rightarrow NM$ is an immersion and this implies that there is a smooth section u of NM in a neighborhood of m such that $\tilde{u} = u \circ \gamma$ in a neighborhood of 0 and $U = u_m$. We deduce that $\tilde{u}' = du \circ \gamma' = D_{\gamma'} u$, so that $V = \tilde{u}'(0) = D_v u$, and $A = \tilde{u}''(0) = D_v D_{\gamma'} u$. Moreover $v^{(2)}(X^2) = v^{(2)}(\tilde{u}''(0)) = (\gamma + u \circ \gamma'')'(0) = 0$. Therefore, X is a 2-critical point of f iff the following conditions are satisfied:

- $v \neq 0$ and $v + D_v u = 0$, or equivalently $v + V = 0$.
- $a + D_v(D_{\gamma'} u) = 0$ or equivalently $a + A = 0$.

The first condition can be separated in two parts. The normal part says that $\nabla_v u = 0$ and the tangent part that $v + (D_v u)^\top = 0$. Multiplying the tangent part by a vector $y \in T_m M$, we obtain $g(v, y) - U \cdot \alpha(v, y) = 0$. In other words, we obtain the following conditions:

- $v \neq 0$ and $g(v, \cdot) - U \cdot \alpha(v, \cdot) = 0$,
- $\nabla_v u = V^\perp = 0$.

Now, we study condition 2). Denote by Y the parallel transport of y along γ . Then, we have $\nabla_{\gamma'} Y = (D_{\gamma'} Y)^\top = 0$, so that

$$Y \cdot D_v(D_{\gamma'} u) = D_v(Y \cdot D_{\gamma'} u) - (D_v Y) \cdot D_v u = -D_v(u \cdot \alpha(Y, \gamma')) + (D_v Y) \cdot v,$$

by 1). Since Y is parallel, the second term vanishes. Having in mind that $\nabla_v u = 0$ by 1.2) and applying 2) we get

$$y \cdot a - U \cdot (\nabla_v \alpha)(y, v) - U \cdot \alpha(\nabla_v Y, v) - U \cdot \alpha(y, a) \quad (6.1)$$

$$= g(a, y) - U \cdot \alpha(a, y) - U \cdot (\nabla_v \alpha)(v, y) = 0, \quad (6.2)$$

i.e.

$$g(a^\top, \cdot) - u \cdot \alpha(a^\top, \cdot) = U \cdot (\nabla_v \alpha)(v, \cdot).$$

Let us put $a^\top = p\nu + b$, with $b \cdot \nu = 0$. Then 2) reads $g(b, \cdot) - U \cdot \alpha(b, \cdot) = U \cdot (\nabla_\nu \alpha)(\nu, \cdot)$ by condition a). If this 1-form acts over ν , and we consider condition a), we obtain:

$$b) \quad U \cdot (\nabla_\nu \alpha)(\nu, \nu) = 0. \quad (6.3)$$

And if that 1-form acts upon the vectors t_j , $j = 2, \dots, k$, we get condition c).

We have $a^\perp = (D_\nu \gamma')^\perp = \alpha(\nu, \nu)$. Hence the normal part of 2) is:

$$2.2) \quad \alpha(\nu, \nu) + A^\perp = 0.$$

Suppose that the system a), b), c) is satisfied by ν, U and $x = x^2 t_2 + \dots + x^k t_k$. It is enough to prove that we may choose γ and \tilde{u} such that $\gamma'(0) = \nu$, $\gamma''(0)^\top = x$, $\tilde{u}(0) = U$, and that 1.2) and 2.2) are satisfied. The conditions on γ can always be satisfied because $\nu \in T_m M$ and the tangent part of $\gamma''(0)$ may be arbitrary. As for \tilde{u} , we take first a parallel orthonormal reference of NM along γ , (u_1, \dots, u_n) . Thus we can write $u(t) = \sum_i p^i(t) u_i(t)$ with the conditions $\sum_i p^i(0) u_i(0) = u$ and $p^i(0) = 0$, $i = 1, \dots, n$. Then $u'(0) = \sum_i p^i(0) u'_i(0)$. Since the u_i are parallel, this implies that $u'(0)^\perp = 0$, which is condition 1.2).

Condition 2.2) can be written as $u''(0)^\perp = -\alpha(\nu, \nu)$ or equivalently as $u_i(0) \cdot u''(0) = -u_i(0) \cdot \alpha(\nu, \nu)$, $i = 1, \dots, n$. Since we have

$$u''(0) = \sum_i (p^{i''}(0) u_i(0) + p^i(0) u''_i(0)),$$

this condition means that 2.2) is satisfied by choosing

$$p^{i''}(0) = -u_i(0) \cdot \alpha(\nu, \nu) - \sum_j p^j(0) u_i(0) \cdot u''_j(0).$$

Hence, our claim is true. \square

6.1. 2-Critical points of surfaces in \mathbb{R}^4

In this section we study the 2-critical points of the normal map for surfaces in \mathbb{R}^4 . Since it is obvious from the preceding theorem that there is a 2-critical point over (ν, U) with $\nu \neq 0$ iff there is one over $(\nu/|\nu|, u)$ we may assume that $g(\nu, \nu) = 1$. We shall denote by $J : T_m M \rightarrow T_m M$ the rotation of 90° .

Proposition 6.3. *The pair (ν, U) , where $0 \neq \nu \in T_m M$ and $U \in N_m M$, admits a 2-critical point of ν iff the following conditions are satisfied:*

- a) $U \cdot \alpha(\nu, \nu) = \nu \cdot \nu$, $U \cdot \alpha(\nu, J\nu) = 0$;
- b) $U \cdot (\nabla_\nu \alpha)(\nu, \nu) = 0$;
- c) $U \cdot \alpha(J\nu, J\nu) \neq \nu \cdot \nu$ or $U \cdot (\nabla_\nu \alpha)(\nu, J\nu) = 0$.

Proof. A vector $x \in T_m M$ that is orthogonal to ν may be written as $qJ\nu$, for some $q \in \mathbb{R}$. Also J is an isometry, whence $g(J\nu, J\nu) = \nu \cdot \nu$. Thus, the conditions of Theorem 6.2 for (ν, U) are now

- a') $g(\nu, \cdot) - U \cdot \alpha(\nu, \cdot) = 0$;
- b) $U \cdot (\nabla_\nu \alpha)(\nu, \nu) = 0$;
- c') $q(\nu \cdot \nu - U \cdot \alpha(J\nu, J\nu)) = U \cdot (\nabla_\nu \alpha)(\nu, J\nu)$ has a solution for $q \in \mathbb{R}$.

Then, a) is equivalent to a') because $(\nu, J\nu)$ is a basis of $T_m M$. Obviously c) is equivalent to c'). \square

Remark 6.4. Note that conditions a) and b) are the same that characterize the strong principal directions defined by Montaldi in [2]. Note also that c') is true generically, so that generically the question whether a direction ν admits 2-critical points is answered by ascertaining that ν satisfies the equation

$$\det(\alpha(\nu, J\nu), (\nabla_\nu \alpha)(\nu, \nu)) = 0, \quad (6.4)$$

which leads to a polynomial equation of fifth degree. For its effective computation see [3].

Conditions a') or a), mean that U is a focal point corresponding to ν and this can be analyzed easily by using the description of the focal locus by means of the inverted pedal. Let us fix the names of some special points. A point m where the curvature ellipse consists of the origin only is called *planar*. If it reduces to a point it is called *umbilic*. If the ellipse lies in a line passing by the origin, it is called *of inflection*. If it is a segment, *semiumbilic*. Note that an inflection point is semiumbilic, an umbilic is of inflection, etc.

For simplicity, in the following we shall assume that v is a unit vector. If m is not semiumbilic, that is its curvature ellipse does not collapse to a segment, then for any v we have $\alpha(v, Jv) \neq 0$. In fact, that vector is the derivative of $\eta(v)$ (its velocity) with respect to t when $v(t) = t_1 \cos t + t_2 \sin t$, where (t_1, t_2) is an orthonormal basis of $T_m M$, and that velocity only vanishes at the cusp points of the ellipse when it degenerates to a segment. Then, a) and b) can be satisfied only when v is a solution of Eq. (6.4). Suppose that then the first alternative of c) does not hold, that is $v \cdot v - U \cdot \alpha(Jv, Jv) = 0$. Then we have $g(Jv, \cdot) - U \cdot \alpha(Jv, \cdot) = 0$, as it is easily verified. But this means that U is the focal point corresponding to v and to Jv , and this means that the tangent line to the ellipse at $\eta(v)$ and $\eta(Jv)$ are the same. Since $\eta(v)$ and $\eta(Jv)$ are opposite with respect to the center of the ellipse, we conclude that the ellipse is a segment against the hypothesis. The conclusion is that *if m is not semiumbilic, then v determines a strong principal direction iff it admits a 2-critical point, or equivalently iff it satisfies Eq. (6.4).*

If m is semiumbilic but not a point of inflection, then for all v such that $\eta(v)$ is not an extremal point of the ellipse we have $\alpha(v, Jv) \neq 0$, so that then there is a solution to a) and b) iff (6.4) is true. But then $\eta(v) = \eta(Jv)$ and $\alpha(v, Jv) = \alpha(Jv, J^2v)$; hence, the point U in \mathcal{F}_m associated to v is the same as that associated to Jv . We conclude that the first alternative of c) is false. Thus, *v admits a 2-critical point iff, in addition to (6.4), the second alternative of c) is true.* If $\eta(v)$ is an extremal point of the ellipse, then $\alpha(v, Jv) = 0$, and $\alpha(v, v) \neq \alpha(Jv, Jv)$ because m is not umbilic. Therefore (v, U) satisfy a) and b) iff $U \cdot (\nabla_v \alpha)(v, v) = 0$ and $U \cdot \alpha(v, v) = 1$. Suppose that the first alternative of c) is false. Then U is a focal point associated both to v and Jv . Therefore, as a consequence of Theorem 4.3, U is exactly the point of intersection of two lines, one being the line orthogonal to $\eta(v)$ passing by the inverse of $\eta(v)$, and the other is the same but relative to $\eta(Jv)$. If then the second alternative of c) does not hold, v does not admit a 2-critical point.

Assume now that m is an inflection non-umbilic point. Then if $\eta(v)$ is not an extreme point, the line tangent to the ellipse at $\eta(v)$ passes by the origin, so that v cannot have an associated focal point, and a fortiori a 2-critical point. If $\eta(v)$ is an extremal point and $\eta(v) = 0$ then it cannot be a strong principal direction nor admit a 2-critical point. If it is extremal but different from the origin, then, as in the case of semiumbilics, it satisfies a) and b) iff $U \cdot (\nabla_v \alpha)(v, v) = 0$ and $U \cdot \alpha(v, v) = 1$. Then, since $\alpha(Jv, Jv)$ is collinear with $\alpha(v, v)$, the first alternative of condition c) only fails if both vectors are equal, that is if the point is umbilic, against the hypothesis. Therefore, *if m is an inflection non-umbilic point, there are no strong principal directions over m , nor 2-critical points over v .*

Suppose that m is umbilic but not planar. Then $\alpha = c \otimes g$, with $0 \neq c \in N_m M$. Thus the vector U must lie in the line orthogonal to c that passes by the inverse of c and be orthogonal to $(\nabla_v \alpha)(v, v)$. The first alternative of c) will fail always, so that (v, U) will admit a 2-critical point if in addition U is orthogonal to $(\nabla_v \alpha)(v, Jv)$, and this requires that this vector and $(\nabla_v \alpha)(v, v)$ be parallel.

Finally, if m is planar, it does not admit strong principal directions nor 2-critical points.

7. 3-Critical points of v for immersions in \mathbb{R}^{k+n}

Now, we will study the 3-critical points of the normal map. In addition to the notation of the preceding section, we will use $\pi_3 : T^3(NM) \rightarrow T^2(NM)$.

Let $X^3 \in T^3(NM)$ be a 3-critical point of v . Then $X^2 = \pi_3(X^3)$ is a 2-critical point, so that we may use the same notation as before plus the following:

1. B will denote $X^3 = \tilde{u}'''(0)$ as element of \mathbb{R}^{k+n} .
2. b will denote $\pi_N^{(3)} X^3 = \gamma'''(0)$ as element of \mathbb{R}^{k+n} .

We will say that v admits a 3-critical point over (v, a, U) , where $v \in T_m M$, $a \in T_m^2 M$ and $U \in N_m M$ if there is a 3-critical point of v , $X^3 \in T^3(NM)$, such that $(\pi_1^3 \circ \pi_N^{(3)})(X^3) = v$, $(\pi_2^3 \circ \pi_N^{(3)})(X^3) = a$ and $\pi_3(X^3) = U$.

Theorem 7.1. *Let $m \in M$, $v \in T_m M$, $a \in T_m^2 M$ and $U \in N_m M$. Let us choose any orthonormal basis of $T_m M$, (t_1, \dots, t_k) such that t_1 and v are parallel. Then, v admits a 3-critical point over (v, a, U) iff the following conditions are satisfied:*

- a) $v \neq 0$ and $g(v, \cdot) - U \cdot \alpha(v, \cdot) = 0$, i.e. U belongs to the focal set over v .
- b) $U \cdot (\nabla_v \alpha)(v, v) = 0$.
- c) $g(x, a) - U \cdot \alpha(x, a) = U \cdot (\nabla_v \alpha)(v, a)$.
- d) $2U \cdot (\nabla_v \alpha)(v, a) + U \cdot (\nabla^2 \alpha)(v, v, v, v) + U \cdot (\nabla_a^\top \alpha)(v, v) = 0$.
- e) The following linear system, whose unknowns are the components of $y = y^2 t_2 + \dots + y^k t_k$ has a solution:

$$g(y, t_j) - U \cdot \alpha(y, t_j) = -\alpha(v, v) \cdot \alpha(v, t_j) + U \cdot (\nabla^2 \alpha)(v, v, v, t_j) \\ + U \cdot (\nabla_a^\top \alpha)(v, t_j) - U \cdot \alpha(\mathcal{A}_{\alpha(v, v)} v, t_j) + 2U \cdot (\nabla_v \alpha)(a^\top, t_j).$$

Proof. Note that c) is the same as Eq. (6.2). Thus, the condition that we must require in addition to a), b) and c) is

$$v^{(3)}(\tilde{u}'''(0)) = (\gamma + u \circ \gamma)'''(0) = b + D_v D_{\gamma'} D_{\gamma'} u = b + B = 0.$$

Let us compute the normal part of $b + B$. If $w \in N_m M$ and W is the parallel transport of w along γ , we will have

$$\begin{aligned} w \cdot b &= w \cdot D_v(D_{\gamma'} \gamma') = D_v(W \cdot D_{\gamma'} \gamma') - D_v W \cdot (D_{\gamma'} \gamma')^\top \\ &= D_v(W \cdot \alpha(\gamma', \gamma')) + w \cdot \alpha(v, a) = w \cdot (\nabla_v \alpha)(v, v) + 3w \cdot \alpha(v, a), \end{aligned}$$

that is we get the equation $B^\perp = -(\nabla_v \alpha)(v, v) - 3\alpha(v, a)$. Now we need to prove that we can set the values of the $p^{i'''}(0)$ in order to satisfy this equation (see the proof of Theorem 6.2), and this is trivial.

As in 6.2, let Y denote the parallel transport of $y \in T_m M$ along γ . Then

$$\begin{aligned} Y \cdot D_{\gamma'} D_{\gamma'} u &= Y \cdot (D_{\gamma'} (D_{\gamma'} u)^\top + D_{\gamma'} (D_{\gamma'} u)^\perp) \\ &= -(D_{\gamma'} u) \cdot \alpha(Y, \gamma') + D_{\gamma'} (Y \cdot D_{\gamma'} u) \\ &= -(D_{\gamma'} u) \cdot \alpha(Y, \gamma') - D_{\gamma'} (u \cdot \alpha(Y, \gamma')) \\ &= -2(D_{\gamma'} u)^\perp \cdot \alpha(Y, \gamma') - u \cdot (\nabla \alpha)(\gamma', Y, \gamma') - u \cdot \alpha(Y, \gamma''), \end{aligned}$$

where, as usual, the tensor field $\nabla \alpha$ over M is defined as $(\nabla \alpha)(x, y, z) = (\nabla_{x^\top} \alpha)(y, z)$, for smooth maps $x, y, z : M \rightarrow \mathbb{R}^{k+n}$. Also, we have $a + A = a + D_v D_{\gamma'} u = 0$. Therefore

$$(D_v Y) \cdot D_v D_{\gamma'} u = -(D_v Y) \cdot a = -(D_v Y) \cdot a^\perp = -\alpha(v, v) \cdot \alpha(v, y).$$

We compute separately the following, having in mind that $a + A = 0$ and $v + V = 0$:

$$\begin{aligned} \nabla_v ((D_{\gamma'} u)^\perp) \cdot \alpha(y, v) &= D_v ((D_{\gamma'} u)^\perp) \cdot \alpha(y, v) \\ &= A \cdot \alpha(y, v) - \alpha(y, v) \cdot \alpha(v, v) = -a \cdot \alpha(y, v) + \alpha(y, v) \cdot \alpha(v, v) = 0. \end{aligned}$$

By substitution and recalling that $\nabla_v Y = 0$ and $V^\perp = 0$ we get

$$\begin{aligned} y \cdot D_v D_{\gamma'} D_{\gamma'} u &= D_v (Y \cdot D_{\gamma'} D_{\gamma'} u) - (D_v Y) \cdot D_v D_{\gamma'} u \\ &= \nabla_v (-2(D_{\gamma'} u)^\perp \cdot \alpha(Y, \gamma') - u \cdot (\nabla \alpha)(\gamma', Y, \gamma') - u \cdot \alpha(Y, \gamma'')) + \alpha(v, v) \cdot \alpha(v, y) \\ &= -U \cdot (\nabla^2 \alpha)(v, v, v, y) - U \cdot (\nabla_a^\top \alpha)(y, v) - 2U \cdot (\nabla_v \alpha)(a, y) \\ &\quad - U \cdot \alpha(y, \nabla_v (\gamma''^\top)) + \alpha(v, v) \cdot \alpha(v, y) \\ &= -U \cdot (\nabla^2 \alpha)(v, v, v, y) - U \cdot (\nabla_a^\top \alpha)(y, v) - 2U \cdot (\nabla_v \alpha)(a, y) \\ &\quad - U \cdot \alpha(y, b - \mathcal{A}_{\alpha(v, v)} v) + \alpha(v, v) \cdot \alpha(v, y). \end{aligned}$$

Hence, from $b + B = 0$ we obtain

$$\begin{aligned} g(b, \cdot) - U \cdot \alpha(b, \cdot) &= -\alpha(v, v) \cdot \alpha(v, \cdot) + U \cdot (\nabla^2 \alpha)(v, v, v, \cdot) \\ &\quad + U \cdot (\nabla_a^\top \alpha)(v, \cdot) + 2U \cdot (\nabla_v \alpha)(a, \cdot) - U \cdot \alpha(\mathcal{A}_{\alpha(v, v)} v, \cdot). \end{aligned}$$

If this 1-form acts upon v we have, taking account of condition a):

$$d) \quad 2U \cdot (\nabla_v \alpha)(v, a) + U \cdot (\nabla^2 \alpha)(v, v, v, v) + U \cdot (\nabla_a^\top \alpha)(v, v) = 0.$$

Then we may find b such that $b + B = 0$ iff there is a vector $y = y^2 t_2 + \dots + y^k t_k$ such that:

$$\begin{aligned} g(y, t_j) - U \cdot \alpha(y, t_j) &= -\alpha(v, v) \cdot \alpha(v, t_j) + U \cdot (\nabla^2 \alpha)(v, v, v, t_j) \\ &\quad + U \cdot (\nabla_a^\top \alpha)(v, t_j) - U \cdot \alpha(\mathcal{A}_{\alpha(v, v)} v, t_j) + 2U \cdot (\nabla_v \alpha)(a^\top, t_j), \end{aligned}$$

which is condition e). \square

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